A CHARACTERIZATION OF THE MURASUGI POLYNOMIAL OF AN EQUIVARIANT SLICE KNOT

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Abstract. We characterize the Murasugi polynomial of an equivariant slice knot by proving a conjecture of J. Davis and S. Naik.

A knot K in S^3 is called *periodic of period* p if there is an orientation preserving action of $\mathbb{Z}/p$ on S^3 which preserves K setwise and its fixed point set A is a circle disjoint to K. A is called the *axis*. Two periodic knots K_0 and K_1 of period p are called *equivariantly concordant* if there is an action of $\mathbb{Z}/p$ on $S^3 \times [0,1]$ such that $K_i \times i$ is periodic via its restriction on $S^3 \times i$, $i = 0, 1$, and there is a locally flat submanifold in $S^3 \times [0,1]$ which is preserved by the action, homeomorphic to $S^1 \times [0,1]$, and bounded by $(K_0 \times 0) \cup -(K_1 \times 1)$. The unknotted circle $S^1 \times 0 \subset \partial(D^2 \times D^2) = S^3$ can be viewed as a periodic knot via the $(2\pi/p)$-rotation on the first D^2 factor. If a periodic knot K is equivariantly concordant to it, K is called an *equivariant slice knot*.

There are several known obstructions for a periodic knot K to being an equivariant slice knot. Some of them are obtained from invariants of K. In [7], Naik used the Alexander polynomial and metabolizers of the Seifert form of K. She also showed that certain Casson-Gordon invariants of K must vanish if K is equivariant slice. In [3], Choi, Ko, and Song defined an obstruction from a Seifert matrix of K.

Further obstructions are obtained by considering the quotient link. Given a periodic knot K with axis A, the orbit space of the $(\mathbb{Z}/p)$-action is again S^3 by the Smith conjecture, and the images $\bar{A}$ and $\bar{K}$ of A and K under the quotient map form a two-component link which is called the *quotient link*. It contains all the essential information on the periodic knot. In [2], Ko and the author developed an obstruction for K to being an equivariant slice knot from knots obtained by surgery on the quotient link. In particular, their Casson-Gordon torsion invariant was used to construct an example of a non-equivariant-slice knot which cannot be detected by other invariants.

Recently, in [4], Davis and Naik have studied the Murasugi polynomial $\Delta_{\mathbb{Z}/p}(g,t)$ of a periodic knot K, which is the image of the Alexander polynomial of the quotient link under the projection $\mathbb{Z}[\mathbb{Z} \times \mathbb{Z}] \to \mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}]$. Here g and t are generators of $\mathbb{Z}/p$ and $\mathbb{Z}$ corresponding to the components $\bar{A}$ and $\bar{K}$, respectively. They proved the following realization theorem of the Murasugi polynomial of an equivariant slice knot:

Theorem 1 (Davis-Naik). *For any $a(g,t) \in \mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}]$ such that $a(g,1) = 1$, there is an equivariant slice knot K with Murasugi polynomial $\Delta_{\mathbb{Z}/p}(g,t) = a(g,t)a(g^{-1},t^{-1})$.*

In fact, their knot K is an equivariant *ribbon* knot, which is a specialization of an equivariant slice knot.

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They conjectured that the converse is true, i.e., the Murasugi polynomial $\Delta_{\mathbb{Z}/p}(g,t)$ of every equivariant slice knot is of the above form. Some related results have been revealed. In [4], by interpreting the Murasugi polynomial as the Reidemeister torsion, Davis and Naik proved that if K an equivariant slice knot, then

$$\Delta_{\mathbb{Z}/p}(g,t)b(g,t)b(g^{-1},t^{-1}) = a(g,t)a(g^{-1},t^{-1})$$

up to $\pm g^r t^s$, for some $a, b \in \mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}]$ such that $a(g,1) = 1 = b(g,1)$. So the question becomes whether $b = 1$ in this result. They also showed that $b = 1$ for an equivariant ribbon knot. In [5], Hillman proved that if K is an equivariant slice knot, then $\Delta_{\mathbb{Z}/p}(g,t) = a(g,t)a(g^{-1},t^{-1})$, up to units, over $\mathbb{Q}[\mathbb{Z}/p \times \mathbb{Z}]$.

The goal of this memo is to prove the Davis-Naik conjecture:

Theorem 2. *If K is an equivariant slice knot, then $\Delta_{\mathbb{Z}/p}(g,t) = a(g,t)a(g^{-1},t^{-1})$, up to $\pm g^r t^s$, for some $a(g,t) \in \mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}]$ such that $a(g,1) = 1$.*

Combined with the above result of Davis and Naik, it characterizes the Murasugi polynomial of an equivariant slice knot.

For the proof of Theorem 2 we use the Blanchfield forms of links which have a well developed theory in the literature. Our arguments are based on the ideas and results of Blanchfield [1], Hillman [5], Levine [6]. We will focus on a special case where we have sharpened versions of well known general results. Theorem 2 will follow from results of this special case.

Let Λ be the group ring $\mathbb{Z}[\mathbb{Z}\times\mathbb{Z}]$ which is identified with the ring of Laurent polynomials in variables x, y, and let Q be its quotient field. For a two-component link L, we denote its exterior by E_L. The abelianization map $\pi_1(E_L) \to \mathbb{Z} \times \mathbb{Z}$ sending meridians to the standard basis gives rise to a Λ-coefficient system on E_L. For a Λ-module M, denote its torsion submodule by tM. A Λ-module is called *pseudozero* if its localization away from π is zero for all prime π in Λ. (Since Λ is a UFD, it agrees with the standard general definition requiring that π is of height one.) We denote by $\hat{t}M$ the quotient of tM by its maximal pseudozero submodule. Then there is a non-degenerated sesquilinear pairing

$$\hat{t}H_1(E_L;\Lambda) \times \hat{t}H_1(E_L, \partial E_L;\Lambda) \to Q/\Lambda$$

due to Blanchfield [1]. (We say $A \times B \to C$ is non-degenerated if the adjoint maps $A \to \operatorname{Hom}(B,C)$ and $B \to \operatorname{Hom}(A,C)$ are injective.) It induces

$$B_L \colon \hat{t}H_1(E_L;\Lambda) \times \hat{t}H_1(E_L;\Lambda) \to Q/\Lambda$$

which is not necessarily non-degenerated. In general, in order to obtain a non-degenerated pairing whose Witt class is a concordance invariant of an arbitrary link, B_L is localized by inverting an appropriate multiplicative subset of Λ (for example, see Hillman [5]).

From now on we assume that a link L has two components with nontrivial linking number. In this special case, we have the crucial advantage that the *unlocalized* Blanchfield form B_L is invariant under concordance.

Lemma 3.

(1) $H_i(\partial E_L;\Lambda) = 0$ *for* $i > 0$.
(2) $H_1(E_L;\Lambda)$ *is a torsion* Λ*-module.*

Proof. (1) follows from that the $(\mathbb{Z}\times\mathbb{Z})$-cover of ∂E_L consists of copies of $\mathbb{R}^2$. (2) is a result of Levine [6, Theorem A, page 378]. See also Hillman [5]. □

From this $H_1(E_L;\Lambda) = tH_1(E_L;\Lambda) = tH_1(E_L,\partial E_L;\Lambda)$ and the above B_L is non-degenerated. Furthermore, a standard argument shows the following result on the unlocalized Blanchfield form:

Lemma 4. *If L and L' are concordant, then $B = B_L \oplus (-B_{L'})$ is metabolic, i.e., there is a submodule P in $\hat{t}H_1(E_L;\Lambda)\oplus\hat{t}H_1(E_{L'};\Lambda)$ such that $P^\perp = P$ with respect to B.*

The argument of Hillman [5, page 37–38] proves this. In fact, in [5], he proved an analogue for the Blanchfield form over a certain localized coefficient system $S^{-1}\Lambda$. The advantage of his localization is that the following fact holds for *any* link: if W is the exterior of a concordance between L and L', then $H_1(\partial W;S^{-1}\Lambda)\cong H_1(E_L;S^{-1}\Lambda)\oplus H_1(E_{L'};S^{-1}\Lambda)$. In our case, it also holds for the (unlocalized) Λ-coefficient homology modules, since $H_1(\partial E_L;\Lambda) = H_1(\partial E_{L'};\Lambda) = 0$ by Lemma 3. From this it can be seen that the argument in [5] works for the unlocalized Blanchfield pairing B_L. We omit the details.

For a finitely generated Λ-module M, $\Delta(M)$ is defined to be the greatest common divisor of $n\times n$ minors of a presentation matrix of M where n is the number of generators of the presentation. It is known that $\Delta(M)$ of the underlying module M of a metabolic non-degenerated pairing is of the form $f(x,y)f(x^{-1},y^{-1})$ (e.g., see [1, 5]). Thus the above lemma implies

$$\Delta(\hat{t}H_1(E_L;\Lambda))\Delta(\hat{t}H_1(E_{L'};\Lambda)) = f(x,y)f(x^{-1},y^{-1})$$

up to units for some $f\in\mathbb{Z}[\mathbb{Z}\times\mathbb{Z}]$. Recall that the Alexander polynomial of L is defined by $\Delta_L(x,y) = \Delta(H_1(E_L;\Lambda))$. Combining Lemma 3 with a result of Blanchfield [1, Theorem 4.7] that $\Delta(tM) = \Delta(\hat{t}M)$, it follows that

$$\Delta(H_1(E_L;\Lambda)) = \Delta(tH_1(E_L;\Lambda)) = \Delta(\hat{t}H_1(E_L;\Lambda)).$$

From this we have

Lemma 5. *If L and L' are concordant, $\Delta_L(x,y)\Delta_{L'}(x,y) = f(x,y)f(x^{-1},y^{-1})$, up to $\pm x^r y^s$, for some $f\in\mathbb{Z}[\mathbb{Z}\times\mathbb{Z}]$.*

If L is a quotient link of an equivariant slice knot, then L is concordant to the Hopf link which has linking number one. (Quotient links of equivariantly concordant periodic knots are concordant in the topologically locally flat category.) Since the Hopf link has trivial Blanchfield form and Alexnader polynomial, we obtain the following consequence:

Proposition 6. *If L is a quotient link of an equivariant slice knot, then*

1. *The Blanchfield form B_L is metabolic.*
2. *The Alexander polynomial $\Delta_L(x,y)$ is of the form $f(x,y)f(x^{-1},y^{-1})$ for some $f\in\mathbb{Z}[\mathbb{Z}\times\mathbb{Z}]$*

In particular, reducing to $\mathbb{Z}[\mathbb{Z}/p\times\mathbb{Z}]$, we have $\Delta_{\mathbb{Z}/p}(g,t) = a(g,t)a(g^{-1},t^{-1})$ over $\mathbb{Z}[\mathbb{Z}/p\times\mathbb{Z}]$ up to $\pm g^r t^s$. Since $\Delta_{\mathbb{Z}/p}(g,1) = 1$, we may assume $a(g,1) = 1$. This proves Theorem 2.

Remark. In the above discussion we may avoid the use of $\hat{t}H_1(E_L;\Lambda)$ since Levine's result in [6] implies that $tH_1(E_L;\Lambda) = \hat{t}H_1(E_L;\Lambda)$ in our case. But we still need to consider $\hat{t}H_1(E_L;\Lambda)$ to prove Lemma 4 using the standard argument as in [5].

Remark. Subsequent to this memo, Davis and Naik have found a different proof of Theorem 2 using the Reidemeister torsion.

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